Milnor-Witt sheaves and Chow-Witt groups

Goal: Quadratic version of Chow groups

§1 Motivation $X \in Sm_k$

Bloch-Kato formula $\boxed{CH^n(X) = H^n(X, \underline{K_n^M})}$

can be computed with Gersten complex

Milnor K-theory: F a field

$K_*^M F := T^* F^\times / J = \bigoplus_{n \geq 0} K_n^M F$

where $T^* F^\times$ = tensor algebra & $J = \{ a \otimes (1-a) \mid a \in F \setminus \{0,1\} \}$

Ex: $\cdot\ K_0^M(F) = \mathbb{Z}$

$\cdot\ K_1^M(F) = F^\times$ $\qquad \{a\} + \{b\} = \{ab\}$

Elements in $K_n^M(F)$: $\quad a_1 \otimes \dots \otimes a_n =: \{a_1, \dots, a_n\}$

v discrete valuation on F
$\mathcal{O}_v$ valuation ring
$\kappa(v)$ residue field
π uniformizer
m_v maximal ideal

$\longrightarrow$ residue homomorphism

$\partial_v : K_*^M(F) \to K_{*-1}^M(\kappa(v))$

st
- $\partial_v(\{\pi, u_1, \dots, u_n\}) = \{\bar{u}_1, \dots, \bar{u}_n\}$ images in $\kappa(v)$
- $\partial_v(\{u_1, \dots, u_n\}) = 0$

$u_1, \dots, u_n \in \mathcal{O}_v^\times$

$X \in Sm_k$

$$K_n^M(X) := \ker\Big(K_n^M(k(X)) \xrightarrow{\text{residue homomorphisms}} \bigoplus_{x \in X^{(1)}} K_{n-1}^M(k(x)) \Big)$$

defines Nisnevich sheaf $\underline{K}_n^M$ on Sm_k

$\exists$ Transfer maps : $F \subset E$ finite field extension

$\rightsquigarrow \mathrm{Tr}_F^E : K_n^M(E) \to K_n^M(F)$

For $n=0$: multiplication by $[E:F]$

$x \in X_{(i)}$, $y \in X_{(i-1)}$ Want $\partial^x_y : K^M_*(k(x)) \to K^M_{*-1}(k(y))$

If $y \notin \overline{\{x\}}$ $\Rightarrow$ set $\partial^x_y = 0$

If $y \in \overline{\{x\}} = Z$, $\tilde{Z} \to Z$ normalization

Set

$$\partial^x_y := \sum_{\substack{z \in \tilde{Z}^{(1)} \\ \text{over } y}} Tr^{k(z)}_{k(y)} \circ \partial_{v_z}$$

For normal $\tilde{Z}$
if $z \in \tilde{Z}^{(1)}$
the local ring
at z is
a valuation
ring

<u>Cohomological cplx</u>: $X \in Sm_k$

$$C^i(X, n) := \bigoplus_{x \in X^{(i)}} K^M_{n-i}(k(x))$$

$$\boxed{CH^n(X) = H^n(C^*(X, n))}$$

$$\dots \to C^{n-1}(X, n) \to C^n(X, n) \to C^{n+1}(X, n) \to \dots$$

$$\dots \to \bigoplus_{x \in X^{(n-1)}} \underbrace{K^M_1(k(x))}_{k(x)^\times} \to \bigoplus_{y \in X^{(n)}} \underbrace{K^M_0(k(y))}_{\mathbb{Z}} \to 0$$

differential $\oplus \partial^x_y$

$\partial^x_y : k(x)^\times \to \mathbb{Z}$

$= \sum [k(z) : k(y)] \cdot ord_{v_z}$

<u>Homological complex</u>: $C_i(X, n) := \bigoplus_{x \in X_{(i)}} K^M_{n+i}(k(x))$

X not necess smooth

$$\boxed{CH_n(X) = H_n(C_*(X, -n))}$$

§2 Milnor-Witt K-theory $\qquad F$ field

$K_*^{MW}(F) := \mathbb{Z}$-graded ring generated by

- $[a]$ $\quad a \in F^*$ in degree $+1$
- η in degree -1

subject to relations

(i) $[a][1-a] = 0 \qquad a \neq 0,1$

(ii) $[ab] = [a] + [b] + \eta[a][b] \qquad a,b \in F^\times$

(iii) $\eta[a] = [a]\eta \qquad a \in F^\times$

(iv) $h := 2 + \eta[-1]$ then $\eta \cdot h = 0$

Notation: $\quad [a_1, \ldots, a_n] := [a_1] \cdot \ldots \cdot [a_n]$

$$\boxed{\begin{array}{ccc} GW(F) & \xrightarrow{\cong} & K_0^{MW}(F) \\ \langle a\rangle & \longmapsto & 1+\eta[a] \end{array}}$$ is a ring homomorphism

I.p. $2+\eta[-1] = 1+\langle -1\rangle = \langle 1\rangle + \langle -1\rangle = h$

- $K_n^{MW}(F) \cong W(F) \quad n<0$
- $K_n^{MW}(F)[\eta^{-1}] = W(F)[\eta, \eta^{-1}]$
- $K_n^{MW}(F)/\eta = K_n^M(F)$
- Have Cartesian square

$\eta^s[a_1,\dots,a_{n+s}] \longmapsto \langle\langle a_1\rangle\rangle \cdot \ldots \cdot \langle\langle a_{n+s}\rangle\rangle$

$$\begin{tikzcd} K_n^{MW}(F) \arrow[r] \arrow[d] & I^n(F) \arrow[d] \\ K_n^M(F) \arrow[r] & I^n(F)/I^{n+1}(F) \end{tikzcd}$$

Here $I(F) \subset GW(F)$
$\qquad \| $
$\quad \ker \mathrm{rk}$

generated by $\langle\langle a\rangle\rangle = \langle a\rangle - 1$

v discrete valuation on F

$\mathcal{O}_v$ valuation ring, m_v maximal ideal

$\kappa(v)$ residue field

π uniformizer

<u>Thm</u> (Morel): $\exists!$ homomorphism of graded groups

$$\partial_v^\pi : K_*^{MW}(F) \to K_{*-1}^{MW}(\kappa(v))$$

- commuting with multiplication by η
- $\partial_v^\pi([\pi, u_1, \dots, u_n]) = [\overline{u}_1, \dots, \overline{u}_n]$

  $\partial_v^\pi([u_1, \dots, u_n]) = 0$ $\qquad u_1, \dots, u_n \in \mathcal{O}_v^\times$

(!) This depends on the choice of uniformizer π:

$\pi' = \pi u \quad u \in \mathcal{O}_v^\times \Rightarrow \quad \partial_v^\pi = \langle u \rangle \partial_v^{\pi'}$

<u>but</u> kernel is independent of the choice of π

$X \in Sm_k$ $\quad K_n^{MW}(X) := \ker \Big(K_n^{MW}(k(X)) \xrightarrow{} \bigoplus_{x \in X^{(1)}} K_n^{MW}(k(x)) \Big)$

residue hom's

This defines a Nisnevich sheaf on Sm_k.

§3 Twisting by a line bundle

To have proper pushforwards, we need to introduce twists by a line bundle.

$Det_{\mathbb{Z}}(X) :=$ category of $\mathbb{Z}$-graded line bundles

scheme

objects: $(i, \mathcal{L})$ $\quad \mathcal{L}$: line bundle $/X$

i: locally constant integer

$$\text{morphism} = \begin{cases} \text{iso's of line bundles} & i = i' \\ \emptyset & \text{otherwise} \end{cases}$$

- has symmetric monoidal structure with unit $(0, \mathcal{O}_X)$

$$D: \text{Vect } X \to Det_{\mathbb{Z}}(X)$$
$$V \to X \longmapsto (\mathrm{rk}\, V, \det V)$$

v.b. $/X$ with only iso's as morphisms

$(i, L) \in \mathrm{Det}_{\mathbb{Z}}(F) \rightsquigarrow L$ is 1-dim F-VS, $i \in \mathbb{Z}$

$\mathbb{Z}[L \setminus \{0\}]$ is a $\mathbb{Z}[F^\times]$ - module

$K_n^{MW}(F)$ is a $\mathbb{Z}[F^\times]$ - module (mult by $\langle u \rangle$)

$$K_n^{MW}(F, (i, L)) := K_n^{MW}(F) \otimes_{\mathbb{Z}[F^\times]} \mathbb{Z}[L \setminus \{0\}]$$

Often we don't need / write the integer i.

$\mathbb{Z}[\mathbb{G}_m] :=$ Nisnevich sheaf associated with $U \mapsto \mathbb{Z}[\mathcal{O}(U)^\times]$

$\mathbb{Z}[\mathcal{L}^\times] :=$ —— " —— $U \mapsto \mathbb{Z}[\mathcal{L}(U)^\times]$

line bundle$/X$

sheaves on X

$$\underline{K_n^{MW}}(i, \mathcal{L}) := \underline{K_n^{MW}} \otimes_{\mathbb{Z}[\mathbb{G}_m]} \mathbb{Z}[\mathcal{L}^\times]$$

sheaf on X

Again often don't write i.

Twisted residue homomorphism

L: loc free rk 1 $\mathcal{O}_v$-module

$$\partial_v : K_*^{MW}(F, (i, L_F)) \to K_{*-1}^{MW}\left(\kappa(v), (-1, (m_v/m_v^2)^\vee) \otimes (i, L_{\kappa(v)})\right)$$

$$\alpha \otimes \ell \longmapsto \partial_v^{\pi}(\alpha) \otimes \bar{\pi}^* \otimes \ell$$

<u>Exercise</u>: This is independent of the choice of uniformizer π.

§ 4 Transfers (k perfect)

$x \in (\mathbb{A}^1_F)^{(1)}$ with max ideal $m_{v_x} \subseteq F[t]$ generated by a irred monic $\pi_x \in F[t]$

Then π_x is a uniformizer for a valuation v_x with residue field $F(x)$.

$\Omega_{E/k}$ = E-module of Kähler differentials, E a fin gen k-algebra

$\omega_{E/k} = \det \Omega_{E/k}$. Have iso $(m_{v_x}/(m_{v_x})^2)^\vee \otimes (\omega_{F[t]/k} \otimes_{F[t]} F(x)) \simeq \omega_{F(x)/k}$

exact sequence

$$m_v/m_v^2 \to \Omega_{F[t]/k} \otimes_{F[t]} F(x) \to \Omega_{F(x)/k} \to 0$$

$L = \omega_{F[t]/k}$

residue hom $\rightsquigarrow$ $\partial_x : K_*^{MW}(F(t), \omega_{F(t)/k}) \to K_{*-1}^{MW}(F(x), \omega_{F(x)/k})$

Thm (Morel): Have a split exact sequence

$$\begin{tikzcd}
0 \arrow[r] & K_*^{MW}(F, \omega_{F/k}) \arrow[r, "\cdot\langle 1\rangle \otimes dt"'] \arrow[drr, bend right, "0"'] & K_*^{MW}(F(t), \omega_{F(t)/k}) \arrow[r] \arrow[d, "-\partial_\infty^{-\frac{1}{t}}"] & \bigoplus\limits_{x \in (\mathbb{A}^1_F)^{(1)}} K_{*-1}^{MW}(F(x), \omega_{F(x)/k}) \arrow[r] \arrow[l, bend right, "s"'] \arrow[dl, "\Sigma \mathrm{Tr}_F^{F(x)}"] & 0 \\
& & K_{*-1}^{MW}(F, \omega_{F/k}) & &
\end{tikzcd}$$

Def (canonical transfer)

$$\mathrm{Tr}_F^{F(p)} : K_{*-1}^{MW}(F(p), \omega_{F(p)/k}) \to \bigoplus_{x \in (\mathbb{A}^1_F)^{(1)}} K_{*-1}^{MW}(F(x), \omega_{F(x)/k}) \xrightarrow{s} K_*^{MW}(F(t), \omega_{F(t)/k}) \xrightarrow{-\partial_\infty^{-\frac{1}{t}}} K_{*-1}^{MW}(F, \omega_{F/k})$$

This is independent of the choice of section s.

In general: For E/F finite, can find $F = F_0 \subset F_1 \subset \dots \subset F_n = E$ st F_i/F_{i-1} are as above. Set $\mathrm{Tr}_F^E := \mathrm{Tr}_F^{F'} \circ \dots \circ \mathrm{Tr}_{F_{n-1}}^E$ ⟸ can show this is well-defined

Ex: $* - 1 \leq 0$ ie $K_*^{MW} = GW$ or W, assume $F(p)/F$ is separable

Then $\mathrm{Tr}_F^{F(p)} : (G)W(F(p), \omega_{F(p)/k}) \to (G)W(F, \omega_{F/k})$

$$\alpha \otimes \ell \longmapsto \mathrm{Tr}_{F(p)/F}(\alpha) \otimes \ell$$

$\Omega_{F(p)/k} \simeq \Omega_{F/k} \otimes_F F(p)$

$\ell \in \omega_{F/k}$ generator

§5 Chow-Witt groups and Rost-Schmid complex

$X \in Sm_k$, $(i, \mathcal{L}) \in Det_{\mathbb{Z}}(X)$ $\qquad Z \subset X$ closed subset

n-th Chow-Witt group $\quad \widetilde{CH}^n_Z(X, (i, \mathcal{L})) := H^n_Z(X, \underline{K}^{MW}_n(i, \mathcal{L}))$

$$\widetilde{CH}^n_Z(X, (i, \mathcal{L})) = \widetilde{CH}^n_Z(X, \mathcal{L})$$

Rost-Schmid complex : X separated of fin type $/k$

Homological complex : $\tilde{C}_j(X, (i, \mathcal{L}), n) := \bigoplus_{x \in X_{(j)}} K^{MW}_{n+j}(k(x), (j, \omega_{k(x)/k}) \otimes (i, \mathcal{L}_x))$

$\widetilde{CH}_n(X, (i, \mathcal{L})) := H_n(\tilde{C}_*(X, (i, \mathcal{L}), -n))$

complex for $\mathcal{L} = \mathcal{O}_X$:

$$\dots \to \tilde{C}_{n+1}(X, -n) \to \tilde{C}_n(X, -n) \to \tilde{C}_{n-1}(X, -n) \to \dots$$

$$\dots \to \bigoplus_{x \in X_{(n+1)}} K^{MW}_1(k(x), \omega_{k(x)/k}) \to \bigoplus_{x \in X_{(n)}} K^{MW}_0(k(x), \omega_{k(x)/k}) \to \bigoplus_{x \in X_{(n-1)}} K^{MW}_{-1}(k(x), \omega_{k(x)/k}) \to \dots$$

$$K^{MW}_0(k(x), \omega_{k(x)/k}) = GW(k(x), \omega_{k(x)/k}), \qquad K^{MW}_{-1}(k(x), \omega_{k(x)/k}) = W(k(x), \omega_{k(x)/k})$$

differential: For $x \in X_{(j)}, y \in X_{(j-1)}$, need $\partial^x_y : K_n^{MW}(k(x), \omega_{k(x)/k}) \to K_{n-1}^{MW}(k(y), \omega_{k(y)/k})$

- $y \notin \overline{\{x\}}$ set $\partial^x_y = 0$
- $y \in \overline{\{x\}} = Z$, $\tilde{Z} \to Z$ normalization

Then

$$\partial^x_y := \sum_{\substack{z \in \tilde{Z}^{(1)} \\ \text{over } y}} \mathrm{Tr}^{k(z)}_{k(y)} \circ \partial_{v_z}$$

$$\begin{tikzcd}
K_n^{MW}(k(x), \omega_{k(x)/k}) \arrow[r, "\partial_{v_z}"] & K_{n-1}^{MW}\Big(k(z), \underbrace{(m_{v_z}/m^2_{v_z})^\vee \otimes \omega_{\tilde{Z}/k}}_{\omega_{k(z)/k}}\Big) \arrow[d, "\mathrm{Tr}^{k(z)}_{k(y)}"] \\
 & K_{n-1}^{MW}(k(y), \omega_{k(y)/k})
\end{tikzcd}$$

$\otimes k(z)$

can remove codim 2 pt $\rightsquigarrow$ assume $\tilde{Z}$ is smooth

$z \xrightarrow{i} \tilde{Z}$

$(m_{v_z}/m^2_{v_z})^2 \to i^*\Omega_{\tilde{Z}/k} \to \omega_{k(z)/k} \to 0$

Cohomological complex

X smooth, connected of dim d_X, for $x \in X^{(j)}$ set $\Lambda(x) := \Lambda^j (m_x / m_x^2)^\vee$

Set $\tilde{C}^j(X, (i, \mathcal{L}), n) := \bigoplus_{x \in X^{(j)}} K^{MW}_{n-j}(k(x), (-j, \Lambda(x)) \otimes (i, \mathcal{L}_x))$

Have iso $(\omega_{X/k}^{-1} \otimes k(x)) \otimes \omega_{k(x)/k} \cong \Lambda(x)$

So $\tilde{C}^j(X, n) = \tilde{C}_{d_X - j}(X, (-d_X, \omega_{X/k}^{-1}), n - d_X)$

Have $\boxed{\widetilde{CH}^n(X, (i, \mathcal{L})) = H^n(C^*(X, (i, \mathcal{L}), n)}$

$$\widetilde{CH}^n(X, (i, \mathcal{L})) = \widetilde{CH}_{d_X - n}(X, (i, \mathcal{L}) \otimes (-d_X, \omega_{X/k}^{-1}))$$

Ex: $\widetilde{CH}^0(\operatorname{Spec} k) = \widetilde{CH}_0(\operatorname{Spec} k) = GW(k)$

§6 Properties $\quad X, Y \in Sm_k$

$f: X \to Y \quad \rightsquigarrow \quad f^*: \widetilde{CH}^n(Y, \mathcal{L}) \to \widetilde{CH}^n(X, f^*\mathcal{L})$

$f: X \to Y$ proper $\rightsquigarrow f_*: \widetilde{CH}^n(X, f^*\mathcal{L} \otimes \omega_{X/k}) \to \widetilde{CH}^{n-d_X+d_Y}(Y, \mathcal{L} \otimes \omega_{Y/k})$

I.p. for $f: X \to \operatorname{Spec} k$ proper

$\rightsquigarrow$ degree map $\quad f_*: \widetilde{CH}^{d_X}(X, \omega_{X/k}) \to GW(k)$

$\underline{K}_n^{MW}(\mathcal{L}) \to \underline{K}_n^M \quad \rightsquigarrow \quad \widetilde{CH}^n(X, \mathcal{L}) \to CH^n(X)$

$\eta \longmapsto 0$

$\underline{K}_n^{MW}(\mathcal{L}) \to W(\mathcal{L}) \quad \rightsquigarrow \quad \widetilde{CH}^n(X, \mathcal{L}) \to H^n(X, W(\mathcal{L}))$

invert η

Have Rost-Schmid complex for this one:
Replace K_n^{MW} by W in complexes above

Homotopy invariance:

$p: V \to X$ v.b $\to$ $p^*: \widetilde{CH}^n(X, \mathcal{L}) \xrightarrow{\cong} \widetilde{CH}^n(V, p^*\mathcal{L})$

Product:

$$\widetilde{CH}^n(X, (i, \mathcal{L})) \times \widetilde{CH}^m(X, (j, \mathcal{N})) \xrightarrow{\boxtimes} \widetilde{CH}^{n+m}(X \times X, (i+j, p_1^*\mathcal{L} \otimes p_2^*\mathcal{N})) \xrightarrow{\Delta^*} \widetilde{CH}^n(X, (i+j, \mathcal{L} \otimes \mathcal{N}))$$

$\langle -1 \rangle^{(n+i)(m+j)}$ -commutative

Projection formula: $f: X \to Y$ proper

Then $f_*(\alpha) \cdot \beta = f_*(\alpha \cdot f^*(\beta))$

$\alpha \in \widetilde{CH}^m(X, \omega_f \otimes f^*\mathcal{L})$

$\beta \in \widetilde{CH}^n(Y, \mathcal{N})$